# Just a Simple Transformation is Enough for Data Protection in Vertical Federated Learning

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MLO Group Meeting, 09.10.2024

# Intro: Privacy & FL

... for Data Protection in Vertical Federated Learning

# Intro: Privacy & FL

- Federated Learning

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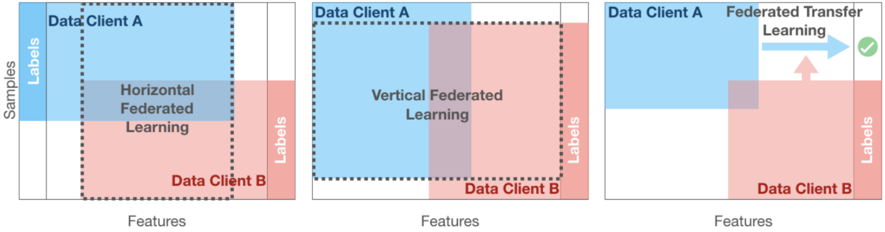


(a) Horizontal Federated Learning (b) Vertical Federated Learning (c) Federated Transfer Learning

# Intro: Privacy & FL

- (Horizontal) Federated Learning

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- (Horizontal) Federated Learning

**Algorithm 1** `FedAvg`

**The $K$ clients are indexed by $k$; $B$ is the local minibatch size, $E$ is the number of local epochs, $\gamma$ is the learning rate, and $C$ is the fraction of clients.**

```
Server executes:
Initialize W_0
for each round t = 1, 2, ... do
    m ← max(C · K, 1)
    S_t ← (random set of m clients)
    for each client k ∈ S_t in parallel do
        W^k_{t+1} ← ClientUpdate(k, W_t)
    end for
    m_t ← Σ_{k∈S_t} n_k
    W_{t+1} ← Σ_{k∈S_t} (n_k / m_t) W^k_{t+1}
end for
```

**ClientUpdate($k$, $W$):** *// Run on client $k$*

```
for each local epoch i from 1 to E do
    for batch b ∈ 𝓑 do
        W ← W − γ∇𝓛(b, W)
    end for
end for
return W to server
```

# Intro: Privacy & FL

- (Horizontal) Federated Learning

**Algorithm 2** `FedAvg`

**The $K$ clients are indexed by $k$; $B$ is the local minibatch size, $E$ is the number of local epochs, $\gamma$ is the learning rate, and $C$ is the fraction of clients.**

**Server executes:**
Initialize $W_0$
**for** each round $t = 1, 2, \dots$ **do**
  $m \leftarrow \max(C \cdot K, 1)$
  $S_t \leftarrow$ (random set of $m$ clients)
  **for** each client $k \in S_t$ **in parallel do**
    $W_{t+1}^k \leftarrow \text{ClientUpdate}(k, W_t)$
  **end for**
    $m_t \leftarrow \sum_{k \in S_t} n_k$
    $W_{t+1} \leftarrow \sum_{k \in S_t} \frac{n_k}{m_t} W_{t+1}^k$
  **end for**

**ClientUpdate($k$, $W$):** *// Run on client $k$*
**for** each local epoch $i$ from 1 to $E$ **do**
  **for** batch $b \in \mathcal{B}$ **do**
    $W \leftarrow W - \gamma \nabla \mathcal{L}(b, W)$
  **end for**
**end for**
**return** $W$ to server

# Intro: Privacy & FL

- (Vertical) Federated Learning

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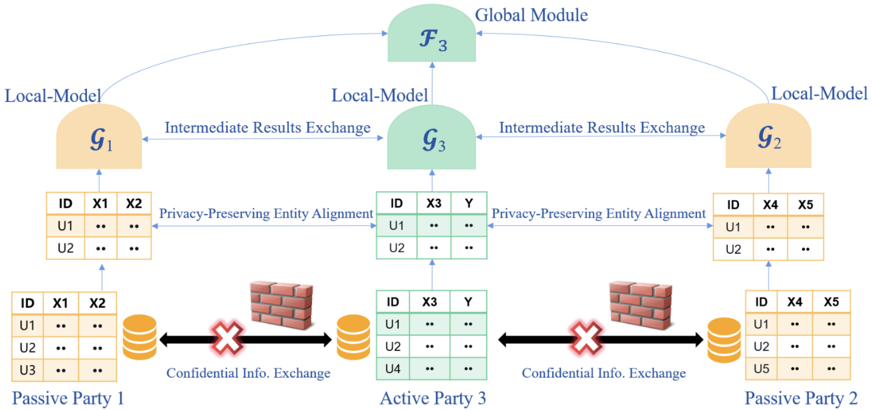
- (Vertical) Federated Learning



# Intro: Privacy & FL

- Vertical Federated Learning → Split Learning

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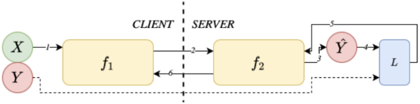


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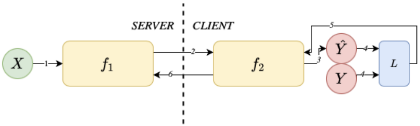
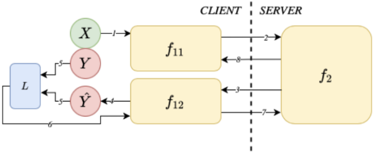


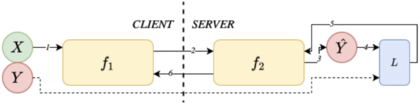

**(b) Training examples and labels are split between the client and the server.**




**(c) Training examples and labels stored only at the client.**

# Intro: Privacy & FL

- Vertical Federated Learning → Split Learning

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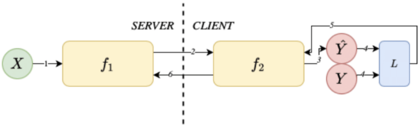


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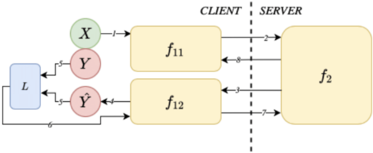


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# Intro: Privacy & FL

- Data Protection

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**Attacks**

Label Inference
Feature Reconstruction
Model Reconstruction

# Intro: Privacy & FL

- Data Protection

| Attacks | Defenses |
| --- | --- |
| Label Inference | Cryptographic-based methods |
| Feature Reconstruction | Differential Privacy |
| Model Reconstruction | Obfuscation-based approaches |

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Feature Reconstruction
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**Attacker's knowledge**

**Defenses**

Cryptographic-based methods
Differential Privacy
Obfuscation-based approaches

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**Attacker's knowledge**
White-Box assumption
Black-Box assumption

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Differential Privacy
Obfuscation-based approaches

**Attacker's acting**

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Honest-but-curious

Malicious

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Feature Reconstruction
Model Reconstruction

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White-Box assumption
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Cryptographic-based methods
Differential Privacy
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# Setup

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- Training under the Split Learning protocol
- Aiming to protect the data against Feature Reconstruction attacks
- While the attacker can be either Malicious or Honest-but-curious

# UnSplit

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Given a client model $f$, its clone $\tilde{f}$ (i.e., the randomly initialized model with the same architecture), the adversary server attempts to solve the two-step optimization problem:

$$\tilde{X}^* = \arg\min_{\tilde{X}} \mathcal{L}_{\mathrm{MSE}}\left(\tilde{f}(\tilde{X}, \tilde{W}),\ f(X, W)\right) + \lambda \mathrm{TV}(\tilde{X}),$$

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$X$, $W$ are the client model's private inputs and parameters; $\mathrm{TV}$ is the total variation distance for image pixels; $\tilde{X}^*$, $\tilde{W}^*$ are the desired variables for the attacker's reconstructed output and parameters

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$$\psi^*_{\mathrm{E}},\ \psi^*_{\mathrm{D}} = \arg\min_{\psi_{\mathrm{E}},\psi_{\mathrm{D}}} \mathcal{L}_{\mathrm{MSE}}\left(\psi_{\mathrm{D}}(\psi_{\mathrm{E}}(X_{\mathrm{pub}})),\ X_{\mathrm{pub}}\right),$$
$$D = \arg\min_{\mathrm{D}} \left[\log(1 - D(\psi_{\mathrm{E}}(X_{\mathrm{pub}}))) + \log(D(f(X)))\right],$$
$$\mathcal{L}^* = \arg\min_{f} \left[\log\left(1 - D(f(X))\right)\right].$$

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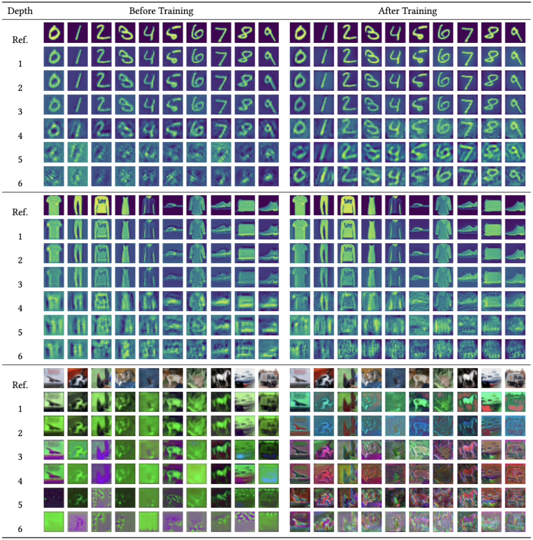
$$\begin{aligned} \psi_{\mathrm{E}}^*,\ \psi_{\mathrm{D}}^* &= \arg\min_{\psi_{\mathrm{E}},\psi_{\mathrm{D}}} \mathcal{L}_{\mathrm{MSE}}\left(\psi_{\mathrm{D}}(\psi_{\mathrm{E}}(X_{\mathrm{pub}})),\ X_{\mathrm{pub}}\right), \\ D &= \arg\min_{\mathrm{D}} \left[\log(1 - D(\psi_{\mathrm{E}}(X_{\mathrm{pub}}))) + \log(D(f(X)))\right], \\ \mathcal{L}^* &= \arg\min_{f} \left[\log\left(1 - D(f(X))\right)\right]. \end{aligned}$$

And, finally, server recovers features with:

$$\tilde{X}^* = \psi_{\mathrm{D}}^*\left(\mathcal{L}^*(X)\right).$$

# Observations 1



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**???**

- Does architectural design play a crucial role in the effectiveness of the latter attacks?
- Is it that simple to attack features, or does the data prior knowledge give a lot?
- Can we develop a theoretical intuition that MLP-based models might be more privacy-preserving againts Feature Reconstruction attacks?

# Observations 3: Theoretical Motivation

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1. **Base case,** $\mathrm{k} = 1$**:** $H_1 = X_1 W_1 = X_1 U U^\top W_1 = \tilde{X}_1 \tilde{W}_1 = \tilde{H}_1$

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1. **Base case,** $\mathrm{k} = 1$**:** $H_1 = X_1 W_1 = X_1 U U^\top W_1 = \tilde{X}_1 \tilde{W}_1 = \tilde{H}_1$
2. **Induction step,** $\mathrm{k} + 1 > 1$**:** Let $H_\mathrm{k} = \tilde{H}_\mathrm{k}$ by induction hypothesis. Then $\partial \mathcal{L} / \partial H_\mathrm{k} = \partial \mathcal{L} / \partial \tilde{H}_\mathrm{k} = G_\mathrm{k} \in \mathbb{R}^{\mathrm{n} \times \mathrm{d_h}}$. Recall, that

$$\frac{\partial \mathcal{L}}{\partial W_\mathrm{k}} = \frac{\partial \mathcal{L}}{\partial H_\mathrm{k}} \frac{\partial H_\mathrm{k}}{\partial W_\mathrm{k}} = X_\mathrm{k}^\top \frac{\partial \mathcal{L}}{\partial H_\mathrm{k}} = X_\mathrm{k}^\top G_\mathrm{k}.$$

# Observations 3: Theoretical Motivation

Then the step of GD for the pairs $\{\mathcal{X}, W_1\}$ and $\{\tilde{\mathcal{X}}, \tilde{W}_1\}$ returns

$$W_{\mathrm{k}+1} = W_{\mathrm{k}} - \gamma X_{\mathrm{k}}^{\top} G_{\mathrm{k}}$$

and

$$\tilde{W}_{\mathrm{k}+1} = \tilde{W}_{\mathrm{k}} - \gamma \tilde{X}_{\mathrm{k}}^{\top} G_{\mathrm{k}} = U^{\top} W_{\mathrm{k}} - \gamma U^{\top} X_{\mathrm{k}}^{\top} G_{\mathrm{k}}$$

respectively.
Thus, at $\mathrm{k}+1$ step

$$\begin{aligned} H_{\mathrm{k}+1} &= X_{\mathrm{k}+1} W_{\mathrm{k}+1} = X_{\mathrm{k}+1} W_{\mathrm{k}} - \gamma X_{\mathrm{k}+1} X_{\mathrm{k}}^{\top} G_{\mathrm{k}} = \\ &= X_{\mathrm{k}+1} U U^{\top} W_{\mathrm{k}} - \gamma X_{\mathrm{k}+1} U U^{\top} X_{\mathrm{k}}^{\top} G_{\mathrm{k}} = \\ &= \tilde{X}_{\mathrm{k}+1} \tilde{W}_{\mathrm{k}} - \gamma \tilde{X}_{\mathrm{k}+1} \tilde{X}_{\mathrm{k}}^{\top} G_{\mathrm{k}} = \\ &= \tilde{X}_{\mathrm{k}+1} \tilde{W}_{\mathrm{k}+1} = \tilde{H}_{\mathrm{k}+1}, \end{aligned}$$

i.e., the activations sent to the server are identical for $\{\mathcal{X}, W_1\}$, $\{\tilde{\mathcal{X}}, \tilde{W}_1\}$ pairs.

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**Lemma 1**

For a one-layer linear model trained using `GD` or `SGD`, there exist continually many pairs of client data and weights initialization that produce the same activations at each step.

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**Remark 1**

Under the conditions of Lemma 1, if the server has no prior information about the distribution of $X$, the label party cannot reconstruct initial data $X$ (only up to an arbitrary orthogonal transformation).

What about the Malicious server???

# Observations 4: Theoretical Motivation

**Corollary 1**

Under the conditions of Lemma1, assume that server knows the first layer $W_1$ of $f$, and let this layer be an **invertible matrix**. Then, the label party cannot reconstruct the initial data $X$ (only up to an arbitrary orthogonal transformation).

# Observations 4: Theoretical Motivation

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Client transmits $H_{\mathrm{k+1}} = XW_{\mathrm{k+1}}$

$$W_{\mathrm{k+1}} = W_{\mathrm{k}} - \gamma X^{\top} G_{\mathrm{k}}^{\mathrm{fake}} =$$
$$= \left(W_{\mathrm{k-1}} - \gamma X^{\top} G_{\mathrm{k-1}}^{\mathrm{fake}}\right) - \gamma X^{\top} G_{\mathrm{k}}^{\mathrm{fake}} =$$
$$= \cdots = W_1 - \gamma X^{\top} \left[\sum_{i=1}^{k} G_{\mathrm{i}}^{\mathrm{fake}}\right].$$

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$$H_{\mathrm{k+1}} = XW_{\mathrm{k+1}} = XW_1 - \gamma XX^\top \left[ \sum_{i=1}^{k} G_{\mathrm{i}}^{\mathrm{fake}} \right],$$

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$$H_{\mathrm{k+1}} = XW_{\mathrm{k+1}} = XW_1 - \gamma XX^\top \left[\sum_{i=1}^{k} G_{\mathrm{i}}^{\mathrm{fake}}\right],$$

$$\tilde{H}_{\mathrm{k+1}} = \tilde{X}W_1 - \gamma \tilde{X}\tilde{X}^\top \left[\sum_{i=1}^{k} G_{\mathrm{i}}^{\mathrm{fake}}\right] = \tilde{X}W_1 - \gamma XX^\top \left[\sum_{i=1}^{k} G_{\mathrm{i}}^{\mathrm{fake}}\right].$$

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The server can only build its attack based on the knowledge of $\tilde{X} = XU$ and $\tilde{X}\tilde{X}^\top$.
This means that it cannot distinguish between two different pairs $\{\mathcal{X}, U\}$ if they generate the same values $\tilde{X}$ and $\tilde{X}\tilde{X}^\top$.

# Observations 4: Theoretical Motivation

**Lemma 2**

Under the conditions of Lemma 1, assume training with the malicious server sending arbitrary vectors instead of real gradients $G = \partial f / \partial H$. In addition, the server knows the initialization of the weight matrix $W_1$. Then, if the client applies a non-trainable orthogonal matrix before $W_1$, the malicious server cannot reconstruct initial data $X$ (only up to an arbitrary orthogonal transformation).

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**Remark 2**

With the same reasons as for Lemma 1, if even the malicious server from Lemma 2 has no prior information about the distribution of $X$, it is impossible for the label party to reconstruct the initial data $X$.

# Cut Layer

Up until now, we considered the client-side model with one linear layer $W$ and proved that orthogonal transformation of data $X$ and weights $W$ lead to the same training protocol

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The intuition behind Lemmas 1 and 2 suggests that in the client model, one should look for layers whose inputs cannot be given the prior distribution.

What about Cut Layer?

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## Cut Layer Lemma

There exist continually many distributions of the activations before the linear Cut Layer that produce the same Split Learning protocol.

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Let us define the client's "previous" parameters (before $W$) as $\theta$ and function of this parameters as $f_\theta : f_\theta(\theta, X) = Z$.

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Then

$$f(X, \theta, W) = H = f_\theta(\theta, X)W = ZW, \; \mathcal{L} = \mathcal{L}(H) = \mathcal{L}(ZW).$$

# Cut Layer

Let us consider the gradient of loss $\mathcal{L}$ w.r.t. $W$ and $\theta$:

$$\frac{\partial \mathcal{L}}{\partial W} = \frac{\partial \mathcal{L}}{\partial H}\frac{\partial H}{\partial W} = Z^\top \frac{\partial \mathcal{L}}{\partial H},$$
$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{\partial \mathcal{L}}{\partial H}\frac{\partial H}{\partial Z}\frac{\partial Z}{\partial \theta} = \left[\frac{\partial Z}{\partial \theta}\right]^* \frac{\partial \mathcal{L}}{\partial H} W^\top = J^* \frac{\partial \mathcal{L}}{\partial H} W^\top,$$

where $J = \frac{\partial Z}{\partial \theta}$ – Jacobian of $f_\theta$.

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where $J = \frac{\partial Z}{\partial \theta}$ – Jacobian of $f_\theta$.
Thus, after the first two iterations we conclude:

$$H_1 = Z_1 W_1, \quad \theta_2 = \theta_1 - \gamma J_1^* \frac{\partial \mathcal{L}}{\partial H_1} W_1^\top, \quad W_2 = W_1 - \gamma Z_1^\top \frac{\partial \mathcal{L}}{\partial H_1},$$

and

$$H_2 = Z_2 W_2 = f_\theta(\theta_2, X) W_2 = f_\theta(\theta_2, X) W_1 - \gamma f_\theta(\theta_2, X) Z_1^\top \frac{\partial \mathcal{L}}{\partial H_1}.$$

# Cut Layer

Adding the additional orthogonal matrix $U$ results in:

$$\tilde{W}_1 = U^\top W_1, \quad \tilde{H}_1 = \tilde{Z}_1 \tilde{W}_1 = (Z_1 U)\tilde{W}_1 = H_1, \quad \tilde{W}_2 = \tilde{W}_1 - \gamma \tilde{Z}_1^\top \frac{\partial \mathcal{L}}{\partial H_1}$$

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$$\begin{aligned}\frac{\partial \tilde{\mathcal{L}}}{\partial \theta_1} &= \frac{\partial \tilde{\mathcal{L}}}{\partial H_1}\frac{\partial H_1}{\partial \tilde{Z}_1}\frac{\partial \tilde{Z}_1}{\partial Z_1}\frac{\partial Z_1}{\partial \theta} = J_1^* \frac{\partial \mathcal{L}}{\partial H_1}\tilde{W}_1^\top U^\top = J_1^* \frac{\partial \mathcal{L}}{\partial H_1}(U^\top W_1)^\top U^\top \\ &= J_1^* \frac{\partial \mathcal{L}}{\partial H_1} W_1^\top = \frac{\partial \mathcal{L}}{\partial \theta_1}.\end{aligned}$$

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Then, for the activations obtained with and without $U$ we claim:

$$\begin{aligned}\tilde{H}_2 &= \tilde{Z}_2 \tilde{W}_2 = f(\theta_2, X) U \tilde{W}_2 \\ &= f(\theta_2, X) U \tilde{W}_1 - \gamma f(\theta_2, X) U \tilde{Z}_1^\top \frac{\partial \mathcal{L}}{\partial H_1} \\ &= H_2.\end{aligned}$$

# Hypothesis

**Hypothesis 1**

*Could it be that the attacks are successful due to the lack of dense layers in the client architecture? Will usage of MLP-based architectures for $f$, instead of CNNs, be more privacy preserving against Model Inversion attack and FSHA?*

# Experiments

Figure: Results of UnSplit attack on CIFAR-10. (**Top**): Original images. (**Middle**): CNN-based client model. (**Bottom**): MLP-Mixer client model.

# Experiments

Figure: Results of UnSplit attack on MNIST. (**Top**): Original images. (**Middle**): CNN-based client model. (**Bottom**): MLP-based client model.

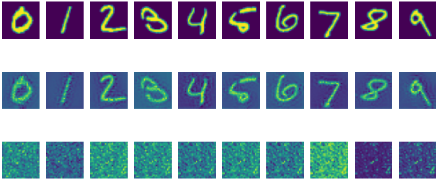
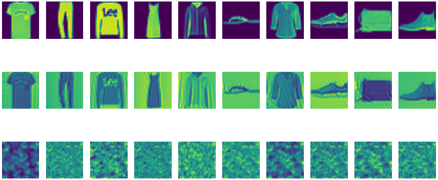


Figure: Results of UnSplit attack on F-MNIST. (**Top**): Original images. (**Middle**): CNN-based client model. (**Bottom**): MLP-based client model.



# Experiments

Figure: Results of FSHA attack on MNIST. (**Top**): Original images. (**Middle**): CNN-based client model. (**Bottom**): MLP-based client model.

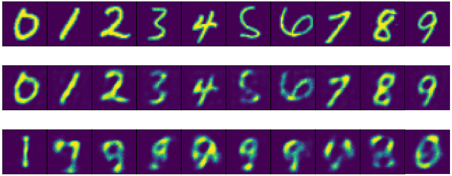
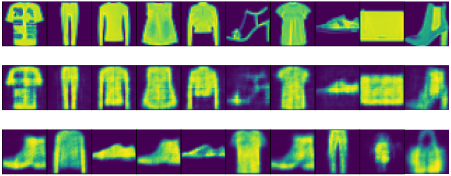


Figure: Results of FSHA attack on F-MNIST. (**Top**): Original images. (**Middle**): CNN-based client model. (**Bottom**): MLP-based client model.



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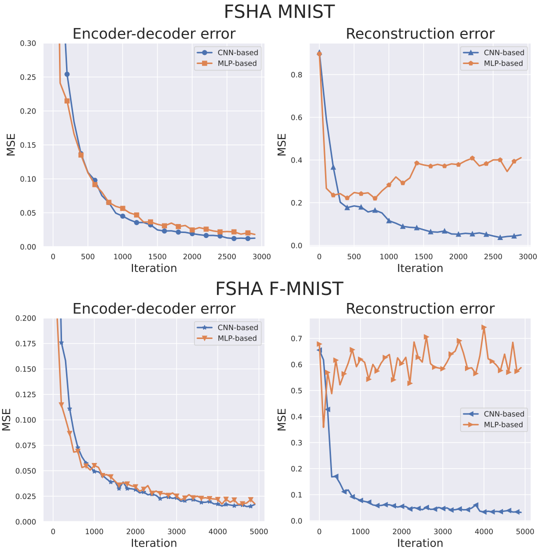

# Split Learning with Adam

Previous theory works only for `(S)GD`-like methods. In practice, all experiments are correct with `Adam` also

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## What about `Adam` in theory?

In `Adam`, the bias-corrected first and second moment estimates, i.e. $m_{\mathrm{k}}$ and $\hat{D}_{\mathrm{k}}$ are:

$$\begin{cases} m_k = \frac{1-\beta_1}{1-\beta_1^{\mathrm{k}}} \sum_{\mathrm{i}=1}^{\mathrm{k}} \beta_1^{\mathrm{k}-\mathrm{i}} \nabla \mathcal{L}(W_{\mathrm{i}}), \\ \hat{D}_{\mathrm{k}}^2 = \frac{1-\beta_2}{1-\beta_2^{\mathrm{k}}} \sum_{\mathrm{i}=1}^{\mathrm{k}} \beta_2^{\mathrm{k}-\mathrm{i}} \,\mathrm{diag}\left(\nabla \mathcal{L}(W_{\mathrm{i}}) \odot \nabla \mathcal{L}(W_{\mathrm{i}})\right), \end{cases}$$

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with the following update rule (without bias-correctness):

$$\begin{cases} \hat{m}_{\mathrm{k}+1} = \beta_1 \hat{m}_\mathrm{k} + (1-\beta_1) \nabla \mathcal{L}(W_\mathrm{k}), \\ \hat{D}_{\mathrm{k}+1}^2 = \beta_2 \hat{D}_\mathrm{k}^2 + (1-\beta_2) \operatorname{diag}(\nabla \mathcal{L}(W_\mathrm{k}) \odot \nabla \mathcal{L}(W_\mathrm{k})). \end{cases}$$

# Split Learning with Adam

**Remark Adam**

Let $\{\tilde{X}, \tilde{W}\} = \{XU, U^\top W\}$ pairs are an orthogonal(semi-orthogonal) transformations of data and weights. Then, these pairs, in general, do not produce the same activations at each step of the Split Learning process with `Adam`.

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Indeed,

$$\hat{\tilde{D}}_{\mathrm{k}}^2 - \beta_2 \hat{D}_{\mathrm{k}-1}^2 = (1 - \beta_2)\,\mathrm{diag}\left(\frac{\partial \mathcal{L}}{\partial \tilde{W}_{\mathrm{k}}} \odot \frac{\partial \mathcal{L}}{\partial \tilde{W}_{\mathrm{k}}}\right)$$

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$$
\begin{aligned}
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&= (1-\beta_2)\,\mathsf{diag}\left(\tilde{X}^\top \frac{\partial \mathcal{L}}{\partial \tilde{H}_{\mathrm{k}}} \odot \tilde{X}^\top \frac{\partial \mathcal{L}}{\partial \tilde{H}_{\mathrm{k}}}\right) \\
&\overset{(\tilde{H}_{\mathrm{k}} = H_{\mathrm{k}})}{=} (1-\beta_2)\,\mathsf{diag}\left(\tilde{X}^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}} \odot \tilde{X}^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}}\right) \\
&= (1-\beta_2)\,\mathsf{diag}\left(U^\top X^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}} \odot U^\top X^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}}\right),
\end{aligned}
$$

# Split Learning with Adam

the similar holds for $\hat{\tilde{m}}_{\mathrm{k}}$

$$\hat{\tilde{m}}_{\mathrm{k}} - \beta_1 \hat{m}_{\mathrm{k}-1} = (1-\beta_1)\frac{\partial \mathcal{L}}{\partial \tilde{W}_{\mathrm{k}}} = (1-\beta_1)\frac{\partial \mathcal{L}}{\partial \tilde{H}_{\mathrm{k}}}\frac{\partial \tilde{H}_{\mathrm{k}}}{\partial \tilde{W}_{\mathrm{k}}} = (1-\beta_1)\tilde{X}^\top \frac{\partial \mathcal{L}}{\partial \tilde{H}_{\mathrm{k}}}$$
$$\overset{(\tilde{H}_{\mathrm{k}}=H_{\mathrm{k}})}{=} (1-\beta_1)\tilde{X}^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}} = (1-\beta_1)U^\top X^\top \frac{\partial \mathcal{L}}{\partial H_{\mathrm{k}}}.$$

# Split Learning with Adam

Then, it is clear how to compare the activations at $\mathrm{k}+1$-th step

$$\begin{cases} \tilde{H}_{\mathrm{k}+1} = \tilde{X}\tilde{W}_{\mathrm{k}+1} = XW_{\mathrm{k}} - \gamma XU\hat{\tilde{D}}_{\mathrm{k}}^{-1}\hat{\tilde{m}}_{\mathrm{k}}, \\ \\ H_{\mathrm{k}+1} = XW_{\mathrm{k}+1} = XW_{\mathrm{k}} - \gamma X\hat{D}_{\mathrm{k}}^{-1}\hat{m}_{\mathrm{k}}. \end{cases}$$

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Therefore, a descrepancy between $\tilde{H}_{\mathrm{k}+1}$ and $H_{\mathrm{k}+1}$ vanishes when

$$U\hat{\tilde{D}}_{\mathrm{k}}^{-1}\hat{\tilde{m}}_{\mathrm{k}} = \hat{D}_{\mathrm{k}}^{-1}\hat{m}_{\mathrm{k}}.$$

# Split Learning with Adam

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**Example** Indeed, let the initial weight and data vectors equal:

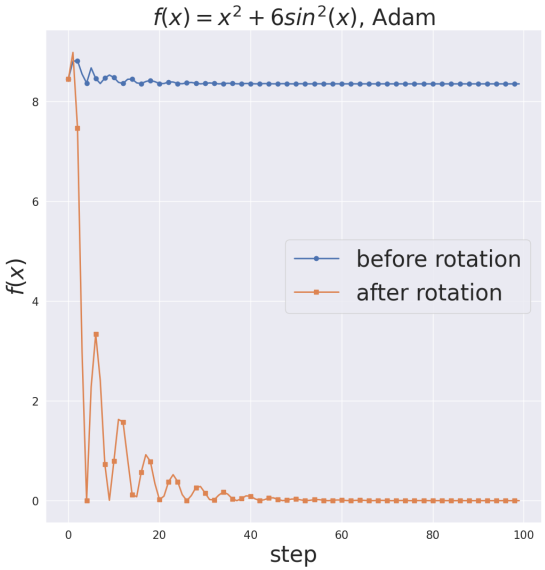
$$w = \left(1.915 + \sqrt{2} \cdot 0.6,\ 0\right)^\top,$$
$$y = (1,\ 0)^\top.$$

We rotate these arguments by an angle of $\frac{\pi}{4}$ with:

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

In addition we pick the learning rate $\gamma = 0.6$. After that, the optimization algorithm stack in the local minima if starting from $(Rw,\ Ry)$ point.

# Split Learning with Adam



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**Remark**

The model's optimal value $\mathcal{L}^*$ after Split Learning is the same for any orthogonal data transformation. Indeed,
$\forall \tilde{X} = XU \; \exists \tilde{W}^* = U^\top W^* : \; \mathcal{L}(\tilde{X}, \tilde{W}^*) = \mathcal{L}^* = \mathcal{L}(X, W^*)$.

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$\forall \tilde{X} = XU \; \exists \tilde{W}^* = U^\top W^* : \; \mathcal{L}(\tilde{X}, \tilde{W}^*) = \mathcal{L}^* = \mathcal{L}(X, W^*)$.

In addition, Split Learning protocol is preserved for PL functions with orthogonal transformations of data and weights

# Split Learning with Adam

## Descent Lemma

Suppose the $L$-smooth Assumption holds for function $\mathcal{L}$. Then we have for all $\mathrm{k} \geq 0$ and $\gamma$, it is true for `Adam` that

$$\mathcal{L}(W_{\mathrm{k}+1}) \leq \mathcal{L}(W_{\mathrm{k}}) + \frac{\gamma}{2\alpha}\|\nabla\mathcal{L}(W_{\mathrm{k}}) - m_{\mathrm{k}}\|^2 - \left(\frac{1}{2\gamma} - \frac{L}{2\alpha}\right)\|W_{\mathrm{k}+1} - W_{\mathrm{k}}\|^2_{\hat{D}_{\mathrm{k}}} - \frac{\gamma}{2}\|\nabla\mathcal{L}(W_{\mathrm{k}})\|^2_{\hat{D}_{\mathrm{k}}^{-1}}.$$

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Without proof here:)

# The End

**Thanks!**